Validity and failure of some entropy inequalities for CAR systems

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Abstract

Basic properties of von Neumann entropy such as the triangle inequality and what we call MONO-SSA are studied for CAR systems. We show that both inequalities hold for every even state by using symmetric purification which is applicable to such a state. We construct a certain class of noneven states giving examples of the nonvalidity of those inequalities.

1 Introduction

Let $\mathcal{H}$ be a Hilbert space and D be a density matrix on $\mathcal{H}$, i.e. a positive trace class operator on $\mathcal{H}$ whose trace is unity. The von Neumann entropy is given by

$$-\mathbf{Tr}\big(D\log D\big), \tag{1}$$

where $\mathbf{Tr}$ denotes the trace which takes the value 1 on each minimal projection. Let ϱ be a normal state of $\mathcal{B}(\mathcal{H})$, the set of all bounded linear operators on $\mathcal{H}$. Then ϱ has its density matrix D_ϱ, and its von Neumann entropy $S(\varrho)$ is given by (1) with $D = D_\varrho$.

It has been known that von Neumann entropy is useful for description and characterization of state correlation for composite systems. Among others, the following inequality called strong subadditivity (SSA) is remarkable:

$$S(\varphi_{\mathrm{I}\cup\mathrm{J}}) - S(\varphi_{\mathrm{I}}) - S(\varphi_{\mathrm{J}}) + S(\varphi_{\mathrm{I}\cap\mathrm{J}}) \leq 0, \tag{2}$$

where I, J, $\mathrm{I}\cap\mathrm{J}$ and $\mathrm{I}\cup\mathrm{J}$ denote the indexes of subsystems and φ_{I} denotes the restriction of a state φ to the subsystem indexed by I, and so on. Such entropy inequalities have been studied for quantum systems, see e.g. Refs. 4, 5, 9, 11, 13, 15, and also their references. However, the composite systems considered there were mostly tensor product of matrix algebras to which we refer as the tensor product systems.

We investigate some well known entropy inequalities, the triangle inequality and MONO-SSA (which will be specified soon), for CAR systems. This study is relevant to our previous works on state correlations such as quantum-entanglement[7] and separability[8] for CAR systems. In a certain sense, the conditions of validity and failure of such entropy inequalities which we are going to establish will explain the similarities and differences in the possible forms of state correlations between CAR and tensor product systems.

Let $\mathcal{L}$ be an arbitrary discrete set. The canonical anticommutation relations (CAR) are

$$\begin{aligned}\{a_i^*, a_j\} &= \delta_{i,j}\,\mathbf{1},\\ \{a_i^*, a_j^*\} &= \{a_i, a_j\} = 0,\end{aligned}$$

where $i, j \in \mathcal{L}$ and $\{A, B\} = AB + BA$ (anticommutator). For each subset I of $\mathcal{L}$, $\mathcal{A}(\mathrm{I})$ denotes the subsystem on I given as the $\mathbf{C}^*$-algebra generated by all a_i^* and a_i with $i \in \mathrm{I}$. For $\mathrm{I}\subset\mathrm{J}$, $\mathcal{A}(\mathrm{I})$ is naturally imbedded in $\mathcal{A}(\mathrm{J})$ as its subalgebra.

We have already shown that SSA (2) holds for the CAR systems. For the convenience, we sketch its proof given in Ref. 6 and Theorem 10.1 of Ref. 2. First we check the commuting square property for a CAR system as follows:

$$\begin{tikzcd} \mathcal{A}(\mathrm{I}\cup\mathrm{J}) \arrow[r, "E_{\mathrm{I}}"] \arrow[d, "E_{\mathrm{J}}"'] & \mathcal{A}(\mathrm{I}) \arrow[d, "E_{\mathrm{I}\cap\mathrm{J}}"] \\ \mathcal{A}(\mathrm{J}) \arrow[r, "E_{\mathrm{I}\cap\mathrm{J}}"'] & \mathcal{A}(\mathrm{I}\cap\mathrm{J}), \end{tikzcd}$$

where E denotes the conditional expectation with respect to the tracial state onto the subsystem with a specified index. From this property SSA follows for every state (without any assumption on the state, like its evenness) by a well-known proof method using the monotonicity of relative entropy under the action of conditional expectations.

We move to entropy inequalities for which CAR makes difference. The following is usually referred to as the triangle inequality:

$$\Big|S(\varphi_{\mathrm{I}}) - S(\varphi_{\mathrm{J}})\Big| \le S(\varphi_{\mathrm{I}\cup\mathrm{J}}), \tag{3}$$

where I and J are disjoint. While this is satisfied for the tensor product systems,[1] it is not valid in general for the CAR systems; there is a counter example.[7]

We next introduce our main target,

$$S(\varphi_{\mathrm{I}}) + S(\varphi_{\mathrm{J}}) \le S(\varphi_{\mathrm{K}\cup\mathrm{I}}) + S(\varphi_{\mathrm{K}\cup\mathrm{J}}), \tag{4}$$

where I, J, and K are disjoint. We may call (4) "MONO-SSA", because it is equivalent to SSA (2) for the tensor product systems at least, and it obviously implies the monotonicity of the following function

$$\mathrm{K} \mapsto S(\varphi_{\mathrm{K}\cup\mathrm{I}}) + S(\varphi_{\mathrm{K}\cup\mathrm{J}}),$$

with respect to the inclusion of the index K. Our question is whether MONO-SSA holds for the CAR systems, if not, under what condition it is satisfied.

The MONO-SSA for the tensor product systems is shown by what is called purification implying the equivalence of MONO-SSA and SSA for those systems (see 3.3 of Ref. 11). We note that the purification is a sort of state extension, and is not automatic for the CAR systems. We shall review the basic concept of state extension.

In the description of a quantum composite system, the total system is given by a $\mathbf{C}^*$-algebra $\mathcal{A}$, and its subsystems are described by $\mathbf{C}^*$-subalgebras $\mathcal{A}_i$ of $\mathcal{A}$ indexed by $i = 1, 2, \cdots$. Let φ be a state of $\mathcal{A}$. We denote its restrictions to $\mathcal{A}_i$ by φ_i. Surely φ_i is a state of $\mathcal{A}_i$. Conversely, suppose that a set of states φ_i of $\mathcal{A}_i$, $i = 1, 2, \cdots$, are given. Then a state φ of $\mathcal{A}$ is called an extension of $\{\varphi_i\}$ if its restriction to each $\mathcal{A}_i$ coincides with φ_i.

For tensor product systems, there always exists a state extension for any given prepared states $\{\varphi_i\}$ on disjoint regions, at least their product state extension $\varphi = \varphi_1 \otimes \cdots \otimes \varphi_i \otimes \cdots$, and generically other extensions. On the contrary, it is not always the case for CAR systems. When two (or more than two) prepared states on disjoint regions are not even, there may be no state extension. We have shown that if all of them are noneven pure states, then there exists no state extension[7].[3]

We explain the above-mentioned purification in terms of state extension. We are given a state ϱ_1 of $\mathcal{A}(\mathrm{I})$. We then prepare some state ϱ_2 on some $\mathcal{A}(\mathrm{J})$ with $\mathrm{J} \cap \mathrm{I} = \emptyset$ such that it has the same nonzero eigenvalues and their multiplicities as ϱ_1 for their density matrices. We want to construct their pure state extension to $\mathcal{A}(\mathrm{I}\cup\mathrm{J})$. We use the term "symmetric purification" to refer to this procedure

where the "symmetric" may indicate the above specified property of ϱ_2. For the tensor product systems, symmetric purification exists for every ϱ_1. On the contrary for the CAR systems, though we can easily make a pure state extension of ϱ_1, its pair ϱ_2 cannot be always chosen among those states which have the same nonzero eigenvalues and their multiplicities as ϱ_1. In the above and what follows, we shall identify states with their density matrices when there is no fear of confusion.

We will show that MONO-SSA is not satisfied in general in § 3. However it is shown to hold for every even state in § 2. **TABLE 1** shows the truth (◯) and the falsity ($\times$) of the entropy inequalities.

TABLE 1. Truth and the falsity of von Neumann entropy inequalities

Property	Tensor-product systems	CAR systems
SSA	◯	◯
Triangle	◯	$\times$ in general, but ◯ for every even state
MONO-SSA	◯	$\times$ in general, but ◯ for every even state

We fix our notation. The even-odd grading Θ is determined by

$$\Theta(a_i^*) = -a_i^*, \quad \Theta(a_i) = -a_i. \tag{5}$$

The even and odd parts of $\mathcal{A}(\mathrm{I})$ are given by

$$\mathcal{A}(\mathrm{I})_{\pm} \equiv \Big\{A \in \mathcal{A}(\mathrm{I}) \,\Big|\, \Theta(A) = \pm A\Big\}.$$

For an element $A \in \mathcal{A}(\mathrm{I})$ we have the decomposition

$$A = A_+ + A_-, \quad A_{\pm} \equiv \frac{1}{2}\big(A \pm \Theta(A)\big) \in \mathcal{A}(\mathrm{I})_{\pm}.$$

For a finite subset I, define

$$v_{\mathrm{I}} \equiv \prod_{i \in \mathrm{I}} v_i, \quad v_i \equiv a_i^* a_i - a_i a_i^*. \tag{6}$$

By a simple computation, v_{I} is a self-adjoint unitary operator in $\mathcal{A}(\mathrm{I})_+$ implementing Θ, namely

$$\mathrm{Ad}(v_{\mathrm{I}})(A) = \Theta(A), \quad A \in \mathcal{A}(\mathrm{I}). \tag{7}$$

For a finite subset I, every even pure state of $\mathcal{A}(\mathrm{I})$ is given by an eigenvector of v_{I} as its vector state.

The following is a simple consequence of the CAR given e.g. in § 4.5 of Ref. 2.

Lemma 1. *Let* I *be a finite subset and* J *be a (finite or infinite) subset disjoint with* J. *Let* $\mathcal{A}(\mathrm{I}' \,|\, \mathrm{I} \cup \mathrm{J}) \equiv \mathcal{A}(\mathrm{I})' \cap \mathcal{A}(\mathrm{I} \cup \mathrm{J})$ *and* $\mathcal{A}(\mathrm{I}' \,|\, \mathrm{J}) \equiv \mathcal{A}(\mathrm{I})' \cap \mathcal{A}(\mathrm{J})$, *the commutant of* $\mathcal{A}(\mathrm{I})$ *in* $\mathcal{A}(\mathrm{I} \cup \mathrm{J})$ *and that in* $\mathcal{A}(\mathrm{J})$, *respectively. Then*

$$\mathcal{A}(\mathrm{I}' \,|\, \mathrm{I} \cup \mathrm{J}) = \mathcal{A}(\mathrm{J})_+ + v_{\mathrm{I}}\, \mathcal{A}(\mathrm{J})_-, \tag{8}$$

$$\mathcal{A}(\mathrm{I}' \,|\, \mathrm{J}) = \mathcal{A}(\mathrm{J})_+. \tag{9}$$

In this note we restrict our discussion to finite-dimensional systems so as to exclude from the outset the cases where our statements themselves on von Neumann entropy do not make sense; for infinite-dimensional systems a density matrix does not exist in general for a given state. (However in the proof of Proposition 8 we shall mention possible infinite-dimensional extensions of some results.)

2 Symmetric Purification for Even States

Symmetric purification is a useful mathematical technique having a lot of applications. For example, we can derive MONO-SSA from SSA for the tensor product systems by using it.

We now discuss symmetric purification for the CAR systems. We shall show its existence for even states.

Lemma 2. *Let* I *and* J *be mutually disjoint finite subsets. Let* ϱ *be an even pure state of* $\mathcal{A}(\mathrm{I}\cup\mathrm{J})$*, and let* ϱ_1 *and* ϱ_2 *be its restrictions to* $\mathcal{A}(\mathrm{I})$ *and* $\mathcal{A}(\mathrm{J})$*. Then the density matrix of* ϱ_1 *has the same nonzero eigenvalues and their multiplicities as those of* ϱ_2*. In particular,* $S(\varrho_1) = S(\varrho_2)$*.*

Proof. We have $\mathcal{A}(\mathrm{I}\cup\mathrm{J}) = \mathcal{A}(\mathrm{I})\otimes\mathcal{A}(\mathrm{I}'\,|\,\mathrm{I}\cup\mathrm{J})$ by (8). By some finite-dimensional Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_{1,2} \equiv \mathcal{H}_1 \otimes \mathcal{H}_2$, we can write $\mathcal{A}(\mathrm{I}\cup\mathrm{J}) = \mathcal{B}(\mathcal{H}_{1,2})$, $\mathcal{A}(\mathrm{I}) = \mathcal{B}(\mathcal{H}_1)$, and $\mathcal{A}(\mathrm{I}'\,|\,\mathrm{I}\cup\mathrm{J}) = \mathcal{B}(\mathcal{H}_2)$.

Since ϱ is a pure state of $\mathcal{A}(\mathrm{I}\cup\mathrm{J})$, its density matrix (with respect to the non-normalized trace **Tr** of $\mathcal{B}(\mathcal{H}_{1,2})$) is a one-dimensional projection operator of $\mathcal{B}(\mathcal{H}_{1,2})$, and hence there exists a unit vector $\xi \in \mathcal{H}_{1,2}$ such that $D_\varrho\eta = (\xi,\eta)\xi$ for any $\eta \in \mathcal{H}_{1,2}$. By using the Schmidt decomposition,[12] we have the following decomposed form:

$$\xi = \sum_i \lambda_i \xi_{1i} \otimes \xi_{2i}, \quad \lambda_i > 0, \tag{10}$$

where $\{\xi_{1i}\}$ and $\{\xi_{2i}\}$ are some orthonormal sets of vectors of $\mathcal{H}_1$ and $\mathcal{H}_2$. For $\nu = 1$, 2, let $P(\xi_{\nu i})$ denote the projection operator on the one-dimensional subspace of $\mathcal{H}_\nu$ containing $\xi_{\nu i}$. We denote the restricted states of ϱ onto $\mathcal{B}(\mathcal{H}_2)$ by $\widetilde{\varrho_2}$. By (10), the density matrices of ϱ_1 and $\widetilde{\varrho_2}$ have the following symmetric forms:

$$D_{\varrho_1} = \sum_i \lambda_i^2 P(\xi_{1i}), \quad D_{\widetilde{\varrho_2}} = \sum_i \lambda_i^2 P(\xi_{2i}). \tag{11}$$

Since ϱ is an even state, its restriction ϱ_2 is even and hence its density matrix D_{ϱ_2} belongs to $\mathcal{A}(\mathrm{J})_+$.

On the other hand, the even state ϱ is invariant under the action of $\mathrm{Ad}(v_{\mathrm{I}\cup\mathrm{J}}) = \mathrm{Ad}(v_{\mathrm{I}}v_{\mathrm{J}})$:

$$v_{\mathrm{I}\cup\mathrm{J}} D_\varrho v_{\mathrm{I}\cup\mathrm{J}} = v_{\mathrm{I}}v_{\mathrm{J}} D_\varrho v_{\mathrm{I}}v_{\mathrm{J}} = D_\varrho.$$

Acting the conditional expectation onto $\mathcal{B}(\mathcal{H}_2)$ with respect to the tracial state of $\mathcal{B}(\mathcal{H}_{1,2})$ on the above equality, we obtain

$$v_{\mathrm{J}} D_{\widetilde{\varrho_2}} v_{\mathrm{J}} = D_{\widetilde{\varrho_2}}$$

noting that v_{I} belongs to $\mathcal{B}(\mathcal{H}_1) = \mathcal{B}(\mathcal{H}_2)' \cap \mathcal{B}(\mathcal{H}_{1,2})$.

We denote $\mathcal{B}(\mathcal{H}_2)_+ \equiv \mathcal{B}(\mathcal{H}_2) \cap \mathcal{A}(\mathrm{I} \cup \mathrm{J})_+$, the set of all invariant elements under $\mathrm{Ad}(v_{\mathrm{I}} v_{\mathrm{J}})$ in $\mathcal{B}(\mathcal{H}_2)$. By (8), $\mathcal{B}(\mathcal{H}_2)_+$ is equal to $\mathcal{A}(\mathrm{J})_+$, and also to the set of all invariant elements under $\mathrm{Ad}(v_{\mathrm{J}})$ in $\mathcal{B}(\mathcal{H}_2)$. Therefore both $D_{\widetilde{\varrho_2}}$ and D_{ϱ_2} belong to $\mathcal{B}(\mathcal{H}_2)_+$. Accordingly, $D_{\widetilde{\varrho_2}}$ is equal to D_{ϱ_2} as the density of the state ϱ restricted to $\mathcal{B}(\mathcal{H}_2)_+$, and hence

$$D_{\varrho_2} = D_{\widetilde{\varrho_2}} = \sum_i \lambda_i^2 P(\xi_{2i}). \tag{12}$$

From (11) and (12), it follows that ϱ_1 and ϱ_2 have the same nonzero eigenvalues and their multiplicities equal to $\{\lambda_i^2\}$. Thus

$$S(\varrho_1) = S(\varrho_2).$$

∎

For a subset I of $\mathcal{L}$, $|\mathrm{I}|$ denotes the number of sites in I.

Lemma 3. *Let* I *be a finite subset and* ϱ_1 *be a state of* $\mathcal{A}(\mathrm{I})$*. Let* J *be a finite subset such that* $\mathrm{J} \cap \mathrm{I} = \emptyset$ *and* $|\mathrm{J}| \geq |\mathrm{I}|$*. Then there exists a pure state* ϱ *on* $\mathcal{A}(\mathrm{I} \cup \mathrm{J})$ *satisfying*

$$\varrho|_{\mathcal{A}(\mathrm{I})} = \varrho_1. \tag{13}$$

Moreover, if ϱ_1 *is even, then the above* ϱ *can be taken to be even.*

Proof. We use the same notation as in the proof of the preceding lemma and write $\mathcal{A}(\mathrm{I} \cup \mathrm{J}) = \mathcal{B}(\mathcal{H}_{1,2})$, $\mathcal{A}(\mathrm{I}) = \mathcal{B}(\mathcal{H}_1)$, and $\mathcal{A}(\mathrm{I}' \,|\, \mathrm{I} \cup \mathrm{J}) = \mathcal{B}(\mathcal{H}_2)$. Let $\varrho_1 = \sum_i \lambda_i^2 P(\xi_{1i})$, where $\lambda_i > 0$, $\{\xi_{1i}\}$ is an orthonormal set of $\mathcal{H}_1$, and $P(\xi_{1i})$ is the projection operator on the one-dimensional subspace of $\mathcal{H}_1$ containing ξ_{1i}. Since $|\mathrm{J}| \geq |\mathrm{I}|$ and hence $\dim \mathcal{H}_2 \geq \dim \mathcal{H}_1$, we can take an orthonormal set of vectors $\{\xi_{2i}\}$ of $\mathcal{H}_2$ having the same cardinality as $\{\xi_{1i}\}$. Define a unit vector $\xi \in \mathcal{H}_{1,2}$ by the same formula as (10) and let ϱ be its vector state, namely the state whose density matrix is the projection operator on the one-dimensional subspace of $\mathcal{H}_{1,2}$ containing ξ. This ϱ is a pure state extension of ϱ_1 to $\mathcal{A}(\mathrm{I} \cup \mathrm{J})$ by its definition.

Assume now that ϱ_1 is even, and hence its density matrix is in $\mathcal{A}(\mathrm{I})_+$. For each eigenvalue, the associated spectral projection is also even and commutes with v_{I}, and its range is invariant under v_{I}. Therefore we can choose an orthonormal basis of the range of the projection which consists of eigenvectors of v_{I}. We take $\{\xi_{1i}\}$ to be a set of eigenvectors of v_{I}.

Since v_{J} belongs to $\mathcal{B}(\mathcal{H}_2)_+ (= \mathcal{A}(\mathrm{J})_+)$, there exists an orthonormal basis of $\mathcal{H}_2$ consisting of eigenvectors of v_{J}. Due to the assumption $|\mathrm{J}| \geq |\mathrm{I}|$, we can take

a set of different eigenvectors $\{\xi_{2i}\}$ of v_{J} such that for each i its eigenvalue, $+1$ or -1, is equal to that of ξ_{1i} for v_{I}. Define a unit vector ξ by (10) using these $\{\xi_{1i}\}$ and $\{\xi_{2i}\}$. Since this ξ is an eigenvector of $v_{\mathrm{I}\cup\mathrm{J}}$ by its definition, its vector state ϱ is even. ∎

Combining the above two lemmas we obtain the following.

Proposition 4. *Let* I *be a finite subset and* ϱ_1 *be an even state of* $\mathcal{A}(\mathrm{I})$*. Let* J *be a finite subset such that* $\mathrm{J}\cap\mathrm{I}=\emptyset$ *and* $|\mathrm{J}|\geq|\mathrm{I}|$*. Then there exists an even pure state* ϱ *on* $\mathcal{A}(\mathrm{I}\cup\mathrm{J})$ *such that its restriction to* $\mathcal{A}(\mathrm{I})$ *is equal to* ϱ_1 *and the density matrix of its restricted state* $\varrho_2 \equiv \varrho|_{\mathcal{A}(\mathrm{J})}$ *has the same nonzero eigenvalues and their multiplicities as those of* ϱ_1*.*

We may call the above state extension from ϱ_1 to ϱ the symmetric purification. Thanks to this, we obtain the following two theorems.

Theorem 5. *Let* I, J *and* K *be mutually disjoint finite subsets. For every even state* φ*, MONO-SSA*

$$S(\varphi_{\mathrm{I}})+S(\varphi_{\mathrm{J}})\leq S(\varphi_{\mathrm{K}\cup\mathrm{I}})+S(\varphi_{\mathrm{K}\cup\mathrm{J}}) \tag{14}$$

is satisfied.

Proof. The equivalence of MONO-SSA and SSA for even states follows from Proposition 4 in the same way as (3) p164 of Ref. 1. Since SSA holds for every state, MONO-SSA is valid for every even state. ∎

Similarly, by using Proposition 4 we immediately obtain the triangle inequality for even states in much the same way as (3.1) of Ref. 1. We omit its proof.

Theorem 6. *Let* I *and* J *be mutually disjoint finite subsets. For every even state* φ*, the triangle inequality*

$$\Big|S(\varphi_{\mathrm{I}})-S(\varphi_{\mathrm{J}})\Big|\leq S(\varphi_{\mathrm{I}\cup\mathrm{J}}) \tag{15}$$

holds.

3 Violation of MONO-SSA

In this section we give a certain class of noneven states.

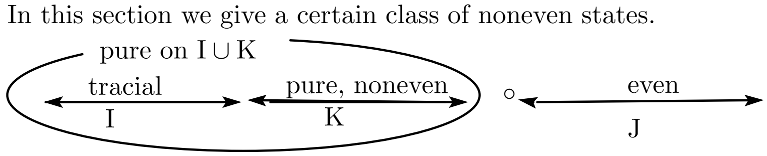

FIG.1. A state not satisfying MONO-SSA.

We shall give a sketch of our model indicated by FIG.1. We can take a pure state $\varrho_{\mathrm{I}\cup\mathrm{K}}$ on $\mathrm{I}\cup\mathrm{K}$ whose restriction ϱ_{K} is a pure state, but ϱ_{I} is non-pure, say the tracial state. Such $\varrho_{\mathrm{I}\cup\mathrm{K}}$ does not satisfy the triangle inequality, because the entropies on I and on K are different, whereas the entropy on $\mathrm{I}\cup\mathrm{K}$ is zero. It can be said that the pure state $\varrho_{\mathrm{I}\cup\mathrm{K}}$ has the asymmetric restrictions in our terminology. This asymmetry is due to the large amount of the oddness of ϱ_{K}, whose precise meaning will be given soon. (Note however that for the infinite-dimensional case, the GNS representations $\pi_{\varrho_{\mathrm{K}}}$ and $\pi_{\varrho_{\mathrm{K}}\Theta}$ should be unitarily equivalent, see Proposition 8 (i).) We take an arbitrary even state ϱ_{J} on J. The desired state on $\varrho_{\mathrm{I}\cup\mathrm{K}\cup\mathrm{J}}$ on $\mathrm{I}\cup\mathrm{K}\cup\mathrm{J}$ is given by the product state extension of $\varrho_{\mathrm{I}\cup\mathrm{K}}$ and ϱ_{J}, which will be denoted by $\varrho_{\mathrm{I}\cup\mathrm{K}}\circ\varrho_{\mathrm{J}}$.

We recall the definition of the transition probability.[14] For two states φ and ψ of $\mathcal{A}(\mathrm{I})$ (where $|\mathrm{I}|$ is finite or infinite), take any representation π of $\mathcal{A}(\mathrm{I})$ on a Hilbert space $\mathcal{H}$ containing vectors Φ and Ψ such that

$$\varphi(A) = (\Phi,\, \pi(A)\Phi), \quad \psi(A) = (\Psi,\, \pi(A)\Psi), \tag{16}$$

for all $A \in \mathcal{A}(\mathrm{I})$. The transition probability between φ and ψ is given by

$$P(\varphi,\, \psi) \equiv \sup |(\Phi,\, \Psi)|^2, \tag{17}$$

where the supremum is taken over all $\mathcal{H}$, π, Φ and Ψ as described above. For a state φ of $\mathcal{A}(\mathrm{I})$, we define

$$p_\Theta(\varphi) \equiv P(\varphi,\, \varphi\Theta)^{1/2}, \tag{18}$$

where $\varphi\Theta$ denotes the state $\varphi\Theta(A) = \varphi(\Theta(A))$, $A \in \mathcal{A}(\mathrm{I})$. Intuitively, $p_\Theta(\varphi)$ quantifies the amount of *oddness* of the state φ. If $p_\Theta(\varphi) = 0$ or nearby, then we may say that the difference between φ and $\varphi\Theta$ is large. If φ is even, $p_\Theta(\varphi)$ takes obviously the maximum value 1.

The following is Lemma 3.1 of Ref. 2.

Lemma 7. *If ϱ_1 is a pure state of $\mathcal{A}(\mathrm{K})$ and π_{ϱ_1} and $\pi_{\varrho_1\Theta}$ are unitarily equivalent, then there exists a self-adjoint unitary $u_1 \in \pi_{\varrho_1}(\mathcal{A}(\mathrm{K})_+)''$ satisfying*

$$u_1\pi_{\varrho_1}(A)u_1 = \pi_{\varrho_1}(\Theta(A)), \quad A \in \mathcal{A}(\mathrm{K}). \tag{19}$$

The next proposition is a basis of our construction. It is a generalization of Ref. 7. The first paragraph is in principle excerpted from Theorem 4 (4) and (5) of Ref. 2. The second paragraph is necessary for the argument of entropy.

Proposition 8. *Let* K *and* I *be mutually disjoint subsets. Assume that ϱ_1 is a (noneven) pure state of $\mathcal{A}(\mathrm{K})$ satisfying $p_\Theta(\varrho_1) = 0$. Assume that ϱ_2 is an even state of $\mathcal{A}(\mathrm{I})$. There exists a joint extension of ϱ_1 and ϱ_2 other than their product state extension if and only if ϱ_1 and ϱ_2 satisfy the following pair of conditions*:

(i) *π_{ϱ_1} and $\pi_{\varrho_1\Theta}$ are unitarily equivalent.*

(ii) *There exists a state $\widetilde{\varrho_2}$ of $\mathcal{A}(\mathrm{I})$ such that $\widetilde{\varrho_2} \neq \widetilde{\varrho_2}\Theta$ and*

$$\varrho_2 = \frac{1}{2}\big(\widetilde{\varrho_2} + \widetilde{\varrho_2}\Theta\big). \tag{20}$$

For each $\widetilde{\varrho_2}$ above, there exists the joint extension of ϱ_1 and ϱ_2 to $\mathcal{A}(\mathrm{K} \cup \mathrm{I})$ denoted by $\psi_{\widetilde{\varrho_2}}$ which satisfies

$$\psi_{\widetilde{\varrho_2}}(A_1 A_2) = \varrho_1(A_1)\varrho_2(A_{2+}) + \overline{\varrho_1}(\pi_{\varrho_1}(A_1)u_1)\widetilde{\varrho_2}(A_{2-}), \tag{21}$$

where $\overline{\varrho_1}$ is the GNS-extension of ϱ_1 to $\pi_{\varrho_1}(\mathcal{A}(\mathrm{K}))''$.

If K and I are finite subsets, then the entropy of $\widetilde{\varrho_2}$ is equal to that of $\psi_{\widetilde{\varrho_2}}$.

Proof. We shall show only the sufficiency of the pair of conditions (i) and (ii). For the necessity of (i) see 5.2 in Ref. 2, and for that of (ii) see (d) in the proof of its Theorem 4(4).

Let $(\mathcal{H}_{\varrho_1}, \pi_{\varrho_1}, \Omega_{\varrho_1})$ be a GNS triplet for ϱ_1 and $(\mathcal{H}_{\widetilde{\varrho_2}}, \pi_{\widetilde{\varrho_2}}, \Omega_{\widetilde{\varrho_2}})$ be that for $\widetilde{\varrho_2}$. Define

$$\begin{aligned} \mathcal{H} &\equiv \mathcal{H}_{\varrho_1} \otimes \mathcal{H}_{\widetilde{\varrho_2}}, \quad \Omega \equiv \Omega_{\varrho_1} \otimes \Omega_{\widetilde{\varrho_2}}, & (22) \\ \pi(A_1 A_2) &\equiv \pi_{\varrho_1}(A_1) \otimes \pi_{\widetilde{\varrho_2}}(A_{2+}) + \pi_{\varrho_1}(A_1)u_1 \otimes \pi_{\widetilde{\varrho_2}}(A_{2-}), & (23) \end{aligned}$$

for $A_1 \in \mathcal{A}(\mathrm{K})$, $A_2 = A_{2+} + A_{2-}$, $A_{2\pm} \in \mathcal{A}(\mathrm{I})_\pm$. Let $\mathbf{1}_1$ be the identity operator of $\mathcal{H}_{\varrho_1}$ and $\mathbf{1}_2$ be that of $\mathcal{H}_{\widetilde{\varrho_2}}$.

We can check that the operators $\pi(\mathcal{A}(\mathrm{K} \cup \mathrm{I}))$ satisfy the CAR by using (19), and hence π extends to a representation of $\mathcal{A}(\mathrm{K} \cup \mathrm{I})$. We define the state $\psi_{\widetilde{\varrho_2}}$ on $\mathcal{A}(\mathrm{K} \cup \mathrm{I})$ as

$$\psi_{\widetilde{\varrho_2}}(A) \equiv (\Omega, \pi(A)\Omega) \tag{24}$$

for $A \in \mathcal{A}(\mathrm{K} \cup \mathrm{I})$. The von Neumann algebra $\pi(\mathcal{A}(\mathrm{K} \cup \mathrm{I}))''$ is generated by $\pi_{\varrho_1}(\mathcal{A}(\mathrm{K}))'' \otimes \mathbf{1}_2$, $\mathbf{1}_1 \otimes \pi_{\widetilde{\varrho_2}}(\mathcal{A}(\mathrm{I})_+)''$, and the weak closure of $\mathbf{1}_1 \otimes \pi_{\widetilde{\varrho_2}}(\mathcal{A}(\mathrm{I})_-)$, where we have noted $u_1 \in \pi_{\varrho_1}(\mathcal{A}(\mathrm{K})_+)'' = \mathcal{B}(\mathcal{H}_{\varrho_1})$. Therefore

$$\pi(\mathcal{A}(\mathrm{K} \cup \mathrm{I}))'' = \mathcal{B}(\mathcal{H}_{\varrho_1}) \otimes \pi_{\widetilde{\varrho_2}}(\mathcal{A}(\mathrm{I}))''. \tag{25}$$

From this it follows that the vector Ω is cyclic for the representation π of $\mathcal{A}(\mathrm{K}\cup\mathrm{I})$ in $\mathcal{H}$. Hence $(\mathcal{H}, \pi, \Omega)$ gives a GNS triplet for the state $\psi_{\widetilde{\varrho_2}}$ on $\mathcal{A}(\mathrm{K} \cup \mathrm{I})$.

We have

$$\psi_{\widetilde{\varrho_2}}(A_1 A_2) = (\Omega, \pi(A_1 A_2)\Omega) = \varrho_1(A_1)\widetilde{\varrho_2}(A_{2+}) + \overline{\varrho_1}(\pi_{\varrho_1}(A_1)u_1)\widetilde{\varrho_2}(A_{2-}), \tag{26}$$

for $A_1 \in \mathcal{A}(\mathrm{K})$, $A_2 = A_{2+} + A_{2-}$, $A_{2\pm} \in \mathcal{A}(\mathrm{I})_\pm$. Taking $A_2 = \mathbf{1}$ in (26), we obtain

$$\psi_{\widetilde{\varrho_2}}(A_1) = \varrho_1(A_1). \tag{27}$$

We will then show

$$\psi_{\widetilde{\varrho_2}}(A_2) = \varrho_2(A_2). \tag{28}$$

Under the condition of Lemma 7 (which is our case), we have

$$p_\Theta(\varrho_1) = |\overline{\varrho_1}(u_1)|, \tag{29}$$

because the transition probability between the vector states of the algebra $\mathcal{B}(\mathcal{H}_{\varrho_1}) = \pi_{\varrho_1}(\mathcal{A}(\mathrm{K}))''$ is equal to the (usual) transition probability of their vectors and hence $p_\Theta(\varrho_1) = |(\Omega_{\varrho_1}, u_1\Omega_{\varrho_1})|$. By the assumption $p_\Theta(\varrho_1) = 0$, (29) implies

$$\overline{\varrho_1}(u_1) = 0. \tag{30}$$

Setting $A_1 = \mathbf{1}$ in (26), we obtain

$$\begin{aligned} \psi_{\widetilde{\varrho_2}}(A_2) &= \widetilde{\varrho_2}(A_{2+}) + \overline{\varrho_1}(u_1)\widetilde{\varrho_2}(A_{2-}) \\ &= \widetilde{\varrho_2}(A_{2+}). \end{aligned} \tag{31}$$

By (20),

$$\varrho_2(A_2) = \varrho_2(A_{2+}) = \widetilde{\varrho_2}(A_{2+}). \tag{32}$$

From (31) and (32), (28) follows. We have now shown that $\psi_{\widetilde{\varrho_2}}$ is an extension of ϱ_1 and ϱ_2.

We will show the second paragraph. By (25) and the commutant theorem, $\pi(\mathcal{A}(\mathrm{K} \cup \mathrm{I}))' = \mathbf{1}_1 \otimes \pi_{\widetilde{\varrho_2}}(\mathcal{A}(\mathrm{I}))'$, where the commutant is taken in each GNS space. Thus we have the following isomorphism:

$$b \mapsto \mathbf{1}_1 \otimes b, \quad b \in \pi_{\widetilde{\varrho_2}}(\mathcal{A}(\mathrm{I}))' \tag{33}$$

from $\pi_{\widetilde{\varrho_2}}(\mathcal{A}(\mathrm{I}))'$ onto $\pi(\mathcal{A}(\mathrm{K} \cup \mathrm{I}))'$. Furthermore by (22) we obtain

$$(\Omega, (\mathbf{1}_1 \otimes b)\Omega) = (\Omega_{\varrho_1} \otimes \Omega_{\widetilde{\varrho_2}}, \Omega_{\varrho_1} \otimes b\Omega_{\widetilde{\varrho_2}}) = (\Omega_{\widetilde{\varrho_2}}, b\Omega_{\widetilde{\varrho_2}}). \tag{34}$$

From the assumption that K and I are finite subsets (which we have not used so far), $\pi_{\widetilde{\varrho_2}}(\mathcal{A}(\mathrm{I}))'$ and $\pi(\mathcal{A}(\mathrm{K} \cup \mathrm{I}))'$ are both finite-dimensional type I factors.

Now we note the following basic fact about GNS representations for states of finite-dimensional type I factors which can be considered as the counterpart of Lemma 2 for the usual case, namely for a pair of isomorphic systems coupled by tensor product. Let ω be a state of a finite-dimensional type I factor $\mathfrak{A}$ and $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ denote a GNS triplet of ω. The GNS vector Ω_ω of ω induces a state on the commutant $\pi_\omega(\mathfrak{A})'$ whose expectation value for $a \in \pi_\omega(\mathfrak{A})'$ is given by $(\Omega_\omega, a\Omega_\omega)$. We call this state on $\pi_\omega(\mathfrak{A})'$ "ω on the commutant". (In our terminology, the pure state with respect to Ω_ω on $\mathcal{B}(\mathcal{H}_\omega) = \pi_\omega(\mathfrak{A}) \otimes \pi_\omega(\mathfrak{A})'$ gives a symmetric purification of ω. Also ω on the commutant is symmetric to ω.) Then the entropy of ω on $\mathfrak{A}$ (equivalently that on $\pi_\omega(\mathfrak{A})$) is equal to the entropy of ω on the commutant by the same reason described in Lemma 2. We note that this holds for a general $\mathbf{C}^*$-algebra if the GNS representation of a given state generates a type I von Neumann algebra with a discrete center. Similarly the extension of Lemma 2 is possible under the above condition on the state.

From the above fact with (33) and (34), we deduce the equality of the entropies of $\widetilde{\varrho_2}$ and of $\psi_{\widetilde{\varrho_2}}$ ∎

Remark 1: Note that this $\psi_{\widetilde{\varrho_2}}$ is a state extension of ϱ_1 and ϱ_2, not that of ϱ_1 and $\widetilde{\varrho_2}$. The possibility of the state extension of ϱ_1 and $\widetilde{\varrho_2}$ is negated by Theorem 4 (3) of Ref. 3.

Remark 2: We can easily make examples of states in this proposition. Take a finite subset K and an odd self-adjoint element A in the algebra $\mathcal{A}(\mathrm{K})$ which we will identify with $\mathcal{B}(\mathcal{H})$ on a finite-dimensional Hilbert space $\mathcal{H}$. Let $\eta \in \mathcal{H}$ be a normalized eigenvector of this A and ω_η denote the associated vector state. Then $\eta \perp v_{\mathrm{K}}\eta$, and $\omega_\eta\Theta$ becomes the vector state with respect to $v_{\mathrm{K}}\eta$. Hence $p_\Theta(\omega_\eta) = 0$. This ω_η obviously satisfies (i). For the existence of ϱ_2 satisfying (ii), take for example the above ω_η for $\widetilde{\varrho_2}$ (with $i \in \mathrm{I}$), $\omega_\eta\Theta$ for $\widetilde{\varrho_2}\Theta$, and their affine sum (20) for ϱ_2.

Proposition 8 yields the following construction giving counter examples of MONO-SSA and those of the triangle inequality.

Theorem 9. *Let* K, I, *and* J *be mutually disjoint subsets. Let* $\varrho_{\mathrm{K}} = \varrho_1$ *and* $\varrho_{\mathrm{I}} = \varrho_2$ *where* ϱ_1 *and* ϱ_2 *are those states on* $\mathcal{A}(\mathrm{K})$ *and on* $\mathcal{A}(\mathrm{I})$ *given in Proposition 8. Let* $\varrho_{\mathrm{K}\cup\mathrm{I}}$ *be the state extension of* ϱ_{K} *and* ϱ_{I} *to* $\mathcal{A}(\mathrm{K}\cup\mathrm{J})$ *given by* $\psi_{\widetilde{\varrho_2}}$ *in the form of (21). Let* ϱ_{J} *be an arbitrary even state of* $\mathcal{A}(\mathrm{J})$*. Then for such* $\varrho_{\mathrm{K}\cup\mathrm{I}}$ *and* ϱ_{J}*, there exists a (unique) product state extension* $\varrho_{\mathrm{K}\cup\mathrm{I}} \circ \varrho_{\mathrm{J}}$ *on* $\mathcal{A}(\mathrm{K}\cup\mathrm{I}\cup\mathrm{J})$*. If all* K, I, *and* J *are finite subsets, then*

$$S(\varrho_{\mathrm{K}\cup\mathrm{I}}) + S(\varrho_{\mathrm{K}\cup\mathrm{J}}) < S(\varrho_{\mathrm{I}}) + S(\varrho_{\mathrm{J}}), \tag{35}$$

and

$$\Big| S(\varrho_{\mathrm{I}}) - S(\varrho_{\mathrm{K}}) \Big| = S(\varrho_{\mathrm{I}}) > S(\varrho_{\mathrm{K}\cup\mathrm{I}}). \tag{36}$$

Proof. By Ref. 10 or Theorem 1 (1) of Ref. 2, there exists a unique product state extension of a pair of states on disjoint regions if (and only if) at least one of them is even, and hence the (unique) existence of $\varrho_{\mathrm{K}\cup\mathrm{I}} \circ \varrho_{\mathrm{J}}$ follows.

By the second paragraph of Proposition 8, we obtain

$$S(\varrho_{\mathrm{K}\cup\mathrm{I}}) = S(\psi_{\widetilde{\varrho_2}}) = S(\widetilde{\varrho_2}). \tag{37}$$

Since $\varrho_2 = \frac{1}{2}\big(\widetilde{\varrho_2} + \widetilde{\varrho_2}\Theta\big)$ and $\widetilde{\varrho_2} \neq \widetilde{\varrho_2}\Theta$, we have

$$S(\varrho_{\mathrm{I}}) = S(\varrho_2) = S\big(1/2(\widetilde{\varrho_2} + \widetilde{\varrho_2}\Theta)\big) > 1/2S(\widetilde{\varrho_2}) + 1/2S(\widetilde{\varrho_2}\Theta) \tag{38}$$

by the strict concavity of von Neumann entropy with respect to the affine sum of states (see *Remark 3* below). Due to the unitary invariance of von Neumann entropy,

$$S(\widetilde{\varrho_2}) = S(\widetilde{\varrho_2}\Theta). \tag{39}$$

By (37), (38) and (39), we have

$$S(\varrho_{\mathrm{I}}) > S(\varrho_{\mathrm{K}\cup\mathrm{I}}). \tag{40}$$

By the product property of $\varrho_{\mathrm{K}\cup\mathrm{J}} = \varrho_{\mathrm{K}} \circ \varrho_{\mathrm{J}}$ the density matrix of $\varrho_{\mathrm{K}\cup\mathrm{J}}$ is a product of the density matrices of ϱ_{K} and ϱ_{J} which mutually commute because $\varrho_{\mathrm{J}} \in \mathcal{A}(\mathrm{J})_{+}$. Hence by a direct computation we have

$$S(\varrho_{\mathrm{K}\cup\mathrm{J}}) = S(\varrho_{\mathrm{K}}) + S(\varrho_{\mathrm{J}}).$$

Since ϱ_{K} is assumed to be pure and hence $S(\varrho_{\mathrm{K}}) = 0$, we have

$$S(\varrho_{\mathrm{K}\cup\mathrm{J}}) = S(\varrho_{\mathrm{J}}). \tag{41}$$

From (40) and (41), we obtain (35). From (40) and $S(\varrho_{\mathrm{K}}) = 0$, we obtain (36). ∎

Remark 3: Let $\mathcal{H}$ be a finite-dimensional Hilbert space. For states φ and ψ on the algebra $\mathcal{B}(\mathcal{H})$ and for $0 \le \lambda \le 1$, the following von Neumann entropy inequalities are well-known:

$$S(\lambda\varphi + (1-\lambda)\psi) \ge \lambda S(\varphi) + (1-\lambda)S(\psi), \tag{42}$$

$$S(\lambda\varphi + (1-\lambda)\psi) \le \lambda S(\varphi) + (1-\lambda)S(\psi) - \lambda\log\lambda - (1-\lambda)\log(1-\lambda). \tag{43}$$

We refer to Proposition 6.2.25 of Ref. 4 for their proofs. We now see the strict concavity of von Neumann entropy which was used in the proof of Theorem 9, namely for $0 < \lambda < 1$ the equality of (42) holds if and only if $\varphi = \psi$. We employ the proof method given in the above-mentioned reference. Let $\mathcal{K}$ be a two-dimensional Hilbert space and P denote a one-dimensional projection of $\mathcal{B}(\mathcal{K})$. We denote $\mathcal{A}_1 \equiv \mathcal{B}(\mathcal{H})$, $\mathcal{A}_2 \equiv \mathcal{B}(\mathcal{K})$, and $\mathcal{A}_{1,2} \equiv \mathcal{B}(\mathcal{H}\otimes\mathcal{K})$. Let ω denote a state on $\mathcal{A}_{1,2}$ whose density matrix D_ω is given by

$$\lambda D_\varphi \otimes P + (1-\lambda)D_\psi \otimes (1-P).$$

We denote ω restricted to $\mathcal{A}_1$ (to $\mathcal{A}_2$, respectively) by ω_1 (and ω_2). We see that ω_1 is equal to $\lambda\varphi + (1-\lambda)\psi$, hence

$$S(\omega_1) = S(\lambda\varphi + (1-\lambda)\psi).$$

Also we have

$$S(\omega) - S(\omega_2) = \lambda S(\varphi) + (1-\lambda)S(\psi).$$

Since

$$S(\omega_1) + S(\omega_2) - S(\omega) = \omega(\log D_\omega - \log(D_{\omega_1} \otimes D_{\omega_2})) \equiv S(\omega|\omega_1\otimes\omega_2) \ge 0,$$

which is equivalent to the subadditivity of entropy, we obtain (42). This $S(\omega|\omega_1\otimes\omega_2)$ is relative entropy of the two states in its argument and is known to have strict positivity, i.e. $S(\omega|\omega_1\otimes\omega_2) = 0$ if and only if $\omega = \omega_1 \otimes \omega_2$, which is equivalent to $\varphi = \psi$ for $0 < \lambda < 1$. Hence our desired strictness of (42) is shown.

Remark 4: We shall give a rough estimation of the amount of violation of the triangle inequality, i.e. $|S(\varphi_{\rm I})-S(\varphi_{\rm K})|-S(\varphi_{\rm K\cup I})$ of (36) for a general state φ. Let $\hat{\varphi}\equiv 1/2(\varphi+\varphi\Theta)$. Then it obviously satisfies the triangle inequality. By (42) and (43), we have $|S(\hat{\varphi}_{\rm K\cup I})-S(\varphi_{\rm K\cup I})|\le -1/2\log 1/2-1/2\log 1/2=\log 2$, and similarly $|S(\hat{\varphi}_{\rm I})-S(\varphi_{\rm I})|\le \log 2$ and $|S(\hat{\varphi}_{\rm K})-S(\varphi_{\rm K})|\le \log 2$. Hence $|S(\varphi_{\rm I})-S(\varphi_{\rm K})|-S(\varphi_{\rm K\cup I})$ is at most $3\log 2$. However, we do not know its possible maximal value. (The violation of the triangle inequality for our concrete model considered in Ref. 7 ranges from 0 up to $\log 2$.)

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